

Quantum logic dequantization: efficiency loss

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Abstract An attempt to ascertain reasons of the quantum logic efficiency was made. To do this, elementary quantum logic operations were reduced to the classical ones via taking the semi-classical limit (dequantization). Calculation of the amount of information lost for any logic operation under the logic reduction was made. General expression for estimation of computational efficiency loss for any dequantized quantum algorithm was derived. Dequantization and estimation of the efficiency loss were demonstrated on quantum discrete fast Fourier transform and Grover search algorithms.

Keywords Quantum logic · Quantum algorithm complexity

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1 Introduction

The problem of quantum computer construction is of current importance in modern physics. The interest to it is caused by achievement of the significant gap in calculation while using quantum algorithms instead of the classical ones. One can rise the question: why is quantum calculus much more efficient than the classical? The wide-spread answer is: the gap is based on the quantum parallelism and, probably, entanglement. But this is only the qualitative explanation. Thus it seems reasonable to try to explain the gap from the basic principles of quantum and classical computation. To do this, formal rules of quantum and classical logics should be stated, as any computation may be interpreted like construction of some logical expression from the elementary ones. These rules were formulated by G.Birkhoff and J.Neumann in their paper [8], where they showed that quantum and classical logics can be extracted from the basic physical principles.

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Since then there was done a lot in this sphere; one can find an overview in [33, 11]. Some of quantum computational structures are presented in [26]. About the investigations in the algebraic structure of logic in the framework of non-commutative geometry one can read [27], that involves Baire*-algebras; in [12] the so-called M-algebras, or measurement algebras, the formalism of which is weaker than that of Hilbert spaces, are being explored. Characterization of the orthomodular lattices via the Sasaki projection one can find in [9]. Papers [34, 13] are devoted to the analysis of contexts, i.e. the maximal sets of commuting logic statements. Different approaches in formal representation and formalism initiation for quantum logic (QL) are being explored: investigations in categorical QL one can find in papers of S. Abramsky and R. Duncan [4] or in S. Abramsky and B. Coecke [2]; about measurement-based QL and computation, that are strongly connected to projective operators logic representation [30], one can read in [6, 29], where the first one is devoted to the “reversible measurement” – a hypothetical operation allowing to “look inside” the quantum computation, and the second one describes measurement-based computation on graph states. Paper [37] develops a theory of finite automata based on QL; in [21] author interprets QL as a language of “pragmatically” decidable assertive formulas, thus formalizing statements about physical quantum systems; paper [25] explores properties of QL representation based on negation and the so-called “and then” connective; in [36] author expands lambda calculus on quantum computation. For some extensions of QL one can look in such issues as [23, 38]; extensions via the addition of modal operator are investigated in [17, 18]. For the computational complexity in quantum and classical logic (CL) calculus one can look in P. Vitanyi [1] or others [3, 28, 10, 20]. Attempts in bridging semantic space and QL one can find in such issues as [32]. Quantum language investigation are being made in such papers as [22]. Basing on these and not only publications one concludes that investigations in logic (especially the quantum one) are rather actual and are interconnected with different spheres of research.

Basing on the calculation gap, one may assume that QL is much more efficient than the classical. Moreover, such assumption is confirmed by the fact that the algebraic structure of QL is constructed with weaker conditions comparatively to the classical case thus allowing wider class of operations on the statements to be processed.

The aim of the paper is to try to ascertain reasons of the QL efficiency and consequently to explain the gap. To do this, we dequantify QL, i.e. show how elementary QL operations reduce to the classical ones when taking the semi-classical limit $\hbar \rightarrow 0$. It was done via the projective operator representation of QL [30]. Basing on it the estimation of the amount of information loss during the logic reduction was made, shedding some light on the loss of logic efficiency and the gap problem. To do this, we used von Neumann and Shannon entropies – for the quantum and dequantized logic gates respectively.

Estimation of the information difference in quantum and classical logic can be made with the help of Kolmogorov complexity. For the different definitions of complexity in quantum case see P. Vitanyi [1] or others [3, 28]; some common properties of them and their possible applications one can find in [10]; also algorithmic entropies can be applied, see [20]. However, the approach presented here having much

in common with the Kolmogorov complexity, differs a bit: we try to estimate the information loss of every elementary logic operation during *reduction* of quantum logic to the classical, then generalizing this to the arbitrary calculation. For this the informational entropy is used. On our opinion, such an approach allows to estimate the contribution to the calculation of any subspace and domain of Hilbert and phase spaces correspondingly.

In [16,15] one can find an extension of [8]. Comparatively to the papers, here we provide dequantization of complete set of logic operations. To do this we use path integral formalism together with Neumann and Shannon entropy definitions for estimation of the efficiency loss. Comparatively to [5], we go further and formalize the approach for any logic gate.

The interrelation between abelian QL subalgebras and CL one can find in [7]. In [14] some aspects of dequantization, which is noted as lifting, for measurement and entanglement were considered with the help of logic entropy. Comparatively to the papers, we consider dequantization of any QL statement using Neumann and Shannon entropy definitions. We demonstrate that the non-commuting propositions play significant role in QL efficiency.

As the next step we develop the algorithm to estimate efficiency loss under dequantization for any QL proposition.

Finally, we exemplify the obtained results with the dequantization of quantum discrete fast Fourier transform (FFT_Q) and Grover search (Gr_Q) algorithms and estimate the efficiency loss for them under the dequantization.

Sections 2 and 3 are devoted to introduction to the classical and quantum logic formalisms correspondingly; for details see [8,30]. The dequantization of QL operations is given in section 4. Estimation of the information loss during the transition from QL to CL is made in section 5. Section 6 presents the theorem which is necessary for the applications of the estimation presented in Section 7. Discussion of the obtained results, their relation to the other complexities and open questions one can find in Section 8.

2 Classical logic

Let Γ_S be the phase space describing physical system S . The state λ of S corresponds to some domain in Γ_S and is characterized by the characteristic function χ_λ defined on Γ_S . The statement “ S possesses physical property λ ”, or “ S is in the state λ ”, will be true for those domains in Γ_S where $\chi_\lambda = 1$ and false where $\chi_\lambda = 0$.

Such characteristic functions may be used to define formal rules and elementary operations of CL on the phase space Γ_S . To do this, one has to describe conjunction, implication, and negation in terms of the phase space subsets; it was done in [8].

Conjunction \wedge may be described as

$$\chi_\wedge = \chi_\lambda \wedge \chi_\mu = \chi_\lambda \chi_\mu \quad (1)$$

and describes the intersection subset.

Implication \leq may be defined as

$$\chi_\lambda \leq \chi_\mu : \chi_\lambda \wedge \chi_\mu = \chi_\lambda \quad (2)$$

and corresponds to rules of the subset inclusion; this operation initiates statement ordering.

Negation \neg

$$\chi_{\neg\lambda} = 1 - \chi_\lambda \quad (3)$$

is equivalent to transition to the complementing subset.

Also the operation of disjunction \vee may be introduced. However, as \vee can be expressed in terms of preliminary operations

$$\chi_\vee = \chi_\lambda + \chi_\mu - \chi_\lambda \chi_\mu, \quad (4)$$

it is not of great importance in the following.

3 Quantum logic

Let H_S be the Hilbert space of physical system S . Let S be in the state $|\zeta\rangle$. Then for any statement about some property λ of S exists projective operator P_λ that projects its state onto the corresponding subspace of H_S . In other words, the statement “ S possesses physical property λ ” will be true if $P_\lambda|\zeta\rangle \neq 0$, and false if $P_\lambda|\zeta\rangle = 0$. It seems that such projective operators in H_S are very similar to the characteristic functions in I_S described above, but there are some significant differences: P_λ defines some subspace in H_S , while χ_λ defines some domain in I_S ; two projective operators do not commute in general, but any two characteristic functions do.

At first let define quantum logical operations for commuting projectors.

Conjunction \wedge is described as

$$P_\wedge|\zeta\rangle = (P_\lambda \wedge P_\mu)|\zeta\rangle = P_\lambda P_\mu|\zeta\rangle = P_\mu P_\lambda|\zeta\rangle, \quad (5)$$

thus describing the intersection of subspaces of *commuting* operators.

Implication \leq is defined as

$$P_\lambda \leq P_\mu: (P_\lambda \wedge P_\mu)|\zeta\rangle = P_\lambda|\zeta\rangle \quad \forall |\zeta\rangle \quad (6)$$

and corresponds to the subspace inclusion; this operation, similarly to the classical case, initiates the statement ordering. As it does not use commutativity in its definition, such form of implication will also hold true for non-commuting projectors.

Negation \neg (complementation) is described as

$$P_{\neg\lambda}|\zeta\rangle = (I - P_\lambda)|\zeta\rangle, \quad (7)$$

where I is the unit operator. This operation is equivalent to transition to the orthogonal subspace.

Conjunction in case of non-commuting operators we define as (see [34], table IV)

$$P_\wedge|\zeta\rangle = (P_\lambda \wedge P_\mu)|\zeta\rangle = \lim_{n \rightarrow \infty} (P_\lambda P_\mu)^n |\zeta\rangle. \quad (8)$$

Such a definition is needed since conjunction leaves the statement belonging to both subspaces only, which are determined by P_λ and P_μ (see [34], the text right after the table). Obviously, if $P_\lambda P_\mu = P_\mu P_\lambda$, then (8) transforms into (5).

As in the CL, disjunction is also expressed in terms of preliminary defined operations, i.e.

$$P_\vee|\zeta\rangle = (P_\lambda + P_\mu - P_\lambda \wedge P_\mu)|\zeta\rangle \quad (9)$$

and thus is not needed in the following.

4 Quantum logic dequantization

Let $|\zeta\rangle$ be any state in Hilbert space H_S of the physical system S . Projective operator P_λ projects the state onto some subspace in H_S ; in the path integral formalism it can be written as

$$P_\lambda|\zeta\rangle = |\lambda\rangle\langle\lambda|\zeta\rangle = \int \mathcal{D}x \mathcal{D}p_x e^{iS_\lambda[x]/\hbar} \int \mathcal{D}y \mathcal{D}p_y e^{iS_{\lambda\rightarrow\zeta}[y]/\hbar}, \quad (10)$$

where integration is made over all phase space trajectories. Here $S_\lambda[x]$ is the action describing transition to the state $|\lambda\rangle$ (that is underlined with a subscript λ) along the fixed phase trajectory x , and $S_{\lambda\rightarrow\zeta}[y]$ describes transition $|\lambda\rangle \rightarrow |\zeta\rangle$ (that is underlined with a subscript $\lambda\rightarrow\zeta$) along the fixed phase trajectory y .

Such a representation of projective operator has much in common with symbol of operator (left or right). It interconnects P_λ (operator) defined in Hilbert space to the action (symbol of operator) defined in phase space.

To get classical physics one has to take the limit $\hbar \rightarrow 0$ thus receiving the classical action. Path integrals extinct when taking the limit because of fast oscillating exponents, and only trajectories for which action has the extremum survive. It gives

$$\begin{aligned} \lim_{\hbar \rightarrow 0} \frac{\hbar}{i} \ln \int \mathcal{D}x \mathcal{D}p_x e^{iS_\lambda[x]/\hbar} \int \mathcal{D}y \mathcal{D}p_y e^{iS_{\lambda\rightarrow\zeta}[y]/\hbar} \\ = \begin{cases} S_\lambda[x] + S_{\lambda\rightarrow\zeta}[y], & \delta S_\lambda[x] = \delta S_{\lambda\rightarrow\zeta}[y] = 0 \\ 0, & \delta S_\lambda[x] \neq 0 \text{ or } \delta S_{\lambda\rightarrow\zeta}[y] \neq 0 \end{cases}, \end{aligned}$$

where δ is variation. So one receives that

$$\lim_{\hbar \rightarrow 0} \frac{\hbar}{i} \ln P_\lambda|\zeta\rangle = (S_\lambda[x] + S_{\lambda\rightarrow\zeta}[y]) \chi_\lambda,$$

where

$$\chi_\lambda = \begin{cases} 1, & \delta S_\lambda[x] = \delta S_{\lambda\rightarrow\zeta}[y] = 0 \\ 0, & \delta S_\lambda[x] \neq 0 \text{ or } \delta S_{\lambda\rightarrow\zeta}[y] \neq 0 \end{cases},$$

or in the compact form

$$P_\lambda|\zeta\rangle \xrightarrow{\hbar \rightarrow 0} \chi_\lambda. \quad (11)$$

Expression (11) defines the transition from projective operator P_λ to some characteristic function χ_λ . The notation χ_λ is used because $|\zeta\rangle$ is any vector from H_S and so there is no need in the subscript ζ . This function defines the classical action that describes transition of the system S from the state with some physical property λ to the state with the property ζ . As one can see, χ_λ vanishes only for those regions of phase space where $\delta(S_\lambda[x] + S_{\lambda\rightarrow\zeta}[y]) \neq 0$.

At first QL operations for commuting projectors will be considered.
Conjunction of two commuting operators

$$(P_\lambda \wedge P_\mu) |\zeta\rangle = |\lambda\rangle \langle \lambda | \mu \rangle \langle \mu | \zeta \rangle$$

after taking the limit $\hbar \rightarrow 0$ (11) transforms as

$$(P_\lambda \wedge P_\mu) |\zeta\rangle \xrightarrow{\hbar \rightarrow 0} \chi_\lambda \chi_\mu, \quad (12)$$

that corresponds to the classical conjunction (1).

Negation (7) can be written as

$$P_{-\lambda} |\zeta\rangle = (I - P_\lambda) |\zeta\rangle = \left(\int_{-\infty}^{+\infty} |\mu\rangle \langle \mu| d\mu - |\lambda\rangle \langle \lambda| \right) |\zeta\rangle,$$

thus giving the equivalent classical expression (3)

$$P_{-\lambda} |\zeta\rangle \xrightarrow{\hbar \rightarrow 0} 1 - \chi_\lambda. \quad (13)$$

Now we will consider the conjunction of non-commuting operators. Such a situation is more complicated because of appearance of the commutator in expressions. Dequantization will consist of two steps: at first any power of product of two non-commuting projective operators will be considered, and only then their conjunction will be dequantized.

Let P_λ, P_μ be two non-commuting projective operators such that

$$P_\lambda P_\mu - P_\mu P_\lambda = i\hbar \Pi, \quad (14)$$

where Π is hermitian. Then, using that

$$\forall k > 0 \quad P_\lambda^k = P_\lambda, \quad P_\mu^k = P_\mu,$$

one obtains

$$\begin{aligned} \forall n > 0 \quad (P_\lambda P_\mu)^n &= (P_\lambda P_\mu)^{n-1} (P_\mu P_\lambda + i\hbar \Pi) = (P_\lambda P_\mu)^{n-1} (P_\lambda + i\hbar \Pi) = \dots \\ &= P_\lambda P_\mu (P_\lambda + i\hbar \Pi)^{n-1}, \end{aligned}$$

where n is integer. Using the following

$$\forall k \geq 0 \quad \begin{cases} P_\lambda (i\hbar \Pi)^{2k} = (i\hbar \Pi)^{2k} P_\lambda \\ P_\lambda (i\hbar \Pi)^{2k+1} = (i\hbar \Pi)^{2k+1} (I - P_\lambda) \end{cases},$$

one obtains then

$$\begin{aligned} \forall k \geq 0 \quad (P_\lambda + i\hbar \Pi)^{2k} &= [P_\lambda + (i\hbar \Pi)^2 + i\hbar \Pi]^k \\ &= \sum_{s=0}^k \frac{k!}{(k-s)!s!} [P_\lambda + (i\hbar \Pi)^2]^s (i\hbar \Pi)^{k-s} \\ &= \sum_{s=0}^k \frac{k!}{(k-s)!s!} \left[P_\lambda \sum_{l=0}^s \frac{s!}{(s-l)!l!} (i\hbar \Pi)^{2(s-l)} + (I - P_\lambda) (i\hbar \Pi)^{2s} \right] (i\hbar \Pi)^{k-s} \\ &= P_\lambda [I + (i\hbar \Pi)^2 + i\hbar \Pi]^k + (I - P_\lambda) (i\hbar \Pi)^k (I + i\hbar \Pi)^k = P_\lambda (I + \alpha)^k + (I - P_\lambda) \alpha^k, \end{aligned}$$

where $\alpha = i\hbar\Pi(I + i\hbar\Pi)$, and finally results in

$$\forall n > 0 \quad (P_\lambda P_\mu)^n = \begin{cases} \beta (I + \alpha)^k + P_\lambda i\hbar\Pi\alpha^k, & n = 2k + 1 \\ \beta [(I + \alpha)^k + i\hbar\Pi\alpha^k] + \gamma_k, & n = 2(k + 1) \end{cases}, \quad (15)$$

where $\beta = P_\mu P_\lambda + (I - P_\lambda) i\hbar\Pi$ and $\gamma_k = P_\lambda (i\hbar\Pi)^2 (I + \alpha)^k$. It gives

$$\forall n > 0 \quad \lim_{\hbar \rightarrow 0} (P_\lambda P_\mu)^n = \lim_{\hbar \rightarrow 0} P_\mu P_\lambda. \quad (16)$$

Using (11) and (16) one gets

$$\begin{aligned} \lim_{\hbar \rightarrow 0} \frac{\hbar}{i} \ln (P_\lambda \wedge P_\mu) |\zeta\rangle &= \lim_{\hbar \rightarrow 0} \frac{\hbar}{i} \ln (P_\mu P_\lambda) |\zeta\rangle \\ &= S_\mu + S_{\mu \rightarrow \lambda} + S_{\lambda \rightarrow \zeta} = S_\lambda + S_{\lambda \rightarrow \mu} + S_{\mu \rightarrow \zeta}, \end{aligned} \quad (17)$$

for which the following variations are true:

$$\begin{aligned} \delta S_\mu &= \delta S_{\mu \rightarrow \lambda} = \delta S_{\lambda \rightarrow \zeta} = 0, \\ \delta S_\lambda &= \delta S_{\lambda \rightarrow \mu} = \delta S_{\mu \rightarrow \zeta} = 0. \end{aligned}$$

Expression (17) determines conjunction dequantization for the non-commuting projectors.

Implication (6) by using previous result also transforms into the classical one (2):

$$P_\lambda \leq P_\mu \xrightarrow{\hbar \rightarrow 0} \chi_\lambda \leq \chi_\mu \quad (18)$$

5 Efficiency loss estimation

Estimating the information, or efficiency, loss during the transition from quantum logic to the classical one may explain the calculation gap. For this purpose informational entropy will be used.

Suppose that system S used for the calculation is in the pure quantum state $|\zeta\rangle$. Neumann entropy of the state H_Q

$$H_Q(|\zeta\rangle) = -\text{Tr} \rho \ln \rho = 0. \quad (19)$$

Here $\rho = |\zeta\rangle\langle\zeta|$ is the density matrix of the system.

After dequantization the description of S is denoted with the corresponding characteristic function χ_λ , see (11), that splits the phase space Γ_S into two domains. So, S can be characterized with Shannon entropy H_C

$$H_C(\chi_\lambda) = -\phi_\lambda \ln \phi_\lambda - (1 - \phi_\lambda) \ln (1 - \phi_\lambda), \quad (20)$$

where

$$\phi_\lambda = \frac{\int \chi_\lambda d\Gamma}{\int d\Gamma}.$$

Thus after dequantization entropy depends on how χ_λ splits I_S ; it is nonzero except when $\phi_\lambda = 0$ or $\phi_\lambda = 1$. As it can be noticed, the entropy is upper bounded, i.e.

$$\forall \lambda \quad H_C(\chi_\lambda) \leq \ln 2. \quad (21)$$

The existence of the upper bound means that some quantum states after the dequantization lose all quantum correlations causing the maximal possible information loss.

Any logic statement consisting of commuting projectors is equivalent to some projector. Consequently, any pure quantum state under the statement transforms to another pure state leaving Neumann entropy H_Q unchanged. However, after statement dequantization the entropy will change because of re-splitting I_S . To show this, the entropy of dequantized logic operations should be explored.

Conjunction entropy of *commuting* projectors after taking the limit $\hbar \rightarrow 0$ (12) is defined as

$$H_C(\chi_\wedge) = -\phi_\wedge \ln \phi_\wedge - (1 - \phi_\wedge) \ln (1 - \phi_\wedge) \leq H_C(\chi_\lambda) + H_C(\chi_\mu), \quad (22)$$

where

$$\phi_\wedge = \frac{\int \chi_\lambda \chi_\mu d\Gamma}{\int d\Gamma}.$$

For the quantum negation entropy after dequantization (13) one receives

$$H_C(\chi_{-\lambda}) = H_C(1 - \chi_\lambda) = H_C(\chi_\lambda). \quad (23)$$

Expression (23) means that because of the symmetry of (20) negation does not change entropy in statement even after the dequantization.

For the implication of commuting projectors one gets that, according to (2) and (6), after the logic conversion (18) entropy will have the following property:

$$P_\lambda \leq P_\mu \quad \Rightarrow \quad H_C(\chi_\wedge) = H_C(\chi_\lambda), \quad (24)$$

where $\chi_\wedge = \chi_\lambda \chi_\mu$.

As before, in case of non-commuting projectors it is enough to consider only entropy of the corresponding conjunction (8) after the logic conversion (17).

Let P_λ, P_μ be two non-commuting projectors that satisfy (14). The initial state $|\zeta\rangle$ of the system can be expanded into series by the eigenstates of commutator Π

$$|\zeta\rangle = \sum_{\pi}^{\dim \Pi} \zeta_\pi |\pi\rangle, \quad \Pi |\pi\rangle = \pi |\pi\rangle.$$

When taking the limit $\hbar \rightarrow 0$ in conjunction (8), the terms containing nonzero powers of Π will vanish in accordance with (15). Thus density matrix ρ should be traced over the eigenstates of Π ; under this averaging pure state transforms into the mixture for which the Neumann entropy is nonzero:

$$H_Q(|\zeta\rangle) \rightarrow H_Q(\rho_\Pi) = -\text{Tr} \rho_\Pi \ln \rho_\Pi = - \sum_{\pi}^{\dim \Pi} |\zeta_\pi|^2 \ln |\zeta_\pi|^2 \leq \ln \dim \Pi. \quad (25)$$

Here $\rho_\Pi = \text{Tr}_\Pi |\zeta\rangle \langle \zeta|$.

In addition, the contribution of every eigenstate $|\pi\rangle$ from the mixture ρ_Π to the whole entropy should be included. Any such term is expressed similarly to (20)

$$H_C(\chi_{\wedge_\Pi|\pi}) = -\phi_{\wedge|\pi} \ln \phi_{\wedge|\pi} - (1 - \phi_{\wedge|\pi}) \ln (1 - \phi_{\wedge|\pi}), \quad (26)$$

where

$$\phi_{\wedge|\pi} = \frac{\int \chi_\lambda \chi_\mu d\Gamma_\pi}{\int d\Gamma_\pi},$$

where χ_{\wedge_Π} is the characteristic function corresponding to the conjunction of our projectors. Here and in the following symbol $|\pi$ means that the transition starting from the state $|\lambda\rangle$ or $|\mu\rangle$ results in the corresponding state $|\pi\rangle$ but not in $|\zeta\rangle$ as before (10).

Summarizing, the whole entropy for the dequantized conjunction of two non-commuting projectors is

$$H_C(\chi_{\wedge_\Pi}) = H_Q(\rho_\Pi) + \sum_{\pi}^{\dim \Pi} |\zeta_\pi|^2 H_C(\chi_{\wedge_\Pi|\pi}). \quad (27)$$

Of course, one can get (22) by formal putting $\dim \Pi = 1$ in (27). The upper bound of this entropy, see (21) and (25), is

$$H_C(\chi_{\wedge_\Pi}) \leq \ln \dim \Pi + \ln 2. \quad (28)$$

Implication of the non-commuting operators is similar to the previous analysis (24): the only difference is that the non-commuting conjunction entropy (27) should be used, i.e.

$$P_\lambda \leq P_\mu \quad \Rightarrow \quad H_C(\chi_{\wedge_\Pi}) = H_C(\chi_\lambda), \quad (29)$$

where projectors satisfy (14). However, this is the generalization of (24); the latter is obtained when putting in (29) $\dim \Pi = 1$ as we did it before.

Obtained results define the entropy of elementary logical statements changing under the logic conversion. Such elementary statements are atomic, and thus are equivalent to the one-qubit register. But for the complete analysis of the information gap registers of arbitrary length should be observed.

Let $|\zeta\rangle^{\otimes N_I}$ be an N_I -qubit register; any calculation with it is similar to construction of some logical expression \mathbb{E}_I from the elementary statements on projectors. Suppose that \mathbb{E}_I has no implications inside (that's underlined with index I) and consists of n_I negations \neg and c_I conjunctions \wedge . The following expression

$$N_I \leq n_I + c_I$$

needs to be true else the such coding can be used where the calculation has no need in $N_I - n_I - c_I$ qubits.

Conjunctions operate the non-commuting projectors in general. Thus, one has to include all commutator (14) contributions while estimating the entropy. After neglecting the first such commutator all subsequent elementary statements will work on the mixture but not on the pure state. However, as negation does not influence the entropy, the conjunctions for the mixture should be observed only.

Suppose that the expression $\mathbb{E}_{I, \Pi_2 \Pi_1}$ consists of two conjunctions characterized with commutators Π_1 (corresponds to the first calculated conjunction) and Π_2 (the second). Entropy for this expression after dequantizing will be

$$H_C(\mathbb{E}_{I, \Pi_2 \Pi_1}|\zeta\rangle) = H_C(\chi_{\wedge \Pi_1}) + \sum_{\pi_1}^{\dim \Pi_1} |\zeta_{\pi_1}|^2 H_C(\chi_{\wedge \Pi_2|\pi_1}).$$

In general, for \mathbb{E}_I on the register $|\zeta\rangle^{\otimes N_I}$ the whole entropy will be estimated by recurrent formula

$$H_C(\mathbb{E}_I|\zeta\rangle^{\otimes N_I}) = \sum_{i=1}^{q_I} H_C(\chi_{\lambda_i}) + H_C(\chi_{\wedge \Pi_1}) + \sum_{\pi_1}^{\dim \Pi_1} |\zeta_{\pi_1}|^2 H_C(\chi_{\wedge \Pi_2|\pi_1}). \quad (30)$$

Here q_I is the number of qubits equipped in no conjunction. Using (21) and (28), one may obtain the upper bound for the entropy:

$$H_C(\mathbb{E}_I|\zeta\rangle^{\otimes N_I}) \leq (q_I + c_I) \ln 2 + \sum_{k=1}^{c_I} \ln \dim \Pi_k. \quad (31)$$

To calculate the entropy for the general expression one must count over all implications made during the calculation. It means that for an expression \mathbb{E} to be calculated on the register $|\zeta\rangle^{\otimes N}$ and containing subexpressions $\{\mathbb{E}_I\}_I$ the entropy $H_C(\mathbb{E}|\zeta\rangle^{\otimes N})$ must consist of contributions from all the subexpressions.

6 Conjunction theorem

Theorem 1 (Conjunction theorem) Let P_λ, P_μ be any two projective operators, $\chi_{\wedge \Pi} = P_\lambda \wedge P_\mu$, $[P_\lambda, P_\mu] = i\hbar \Pi$. Then

$$\forall k > 0, \quad H_C((P_\lambda P_\mu)^k) = H_C(\chi_{\wedge \Pi}). \quad (32)$$

Proof As $\chi_{\wedge \Pi} = \lim_{n \rightarrow \infty} (P_\lambda P_\mu)^n$, one can write $\forall k > 0$

$$\begin{aligned} (P_\lambda P_\mu)^k \chi_{\wedge \Pi} &= \chi_{\wedge \Pi} \Rightarrow [I - (P_\lambda P_\mu)^k] \chi_{\wedge \Pi} = 0, \\ \chi_{\wedge \Pi} (P_\lambda P_\mu)^k &= \chi_{\wedge \Pi} \Rightarrow \chi_{\wedge \Pi} [I - (P_\lambda P_\mu)^k] = 0, \end{aligned}$$

and therefore

$$[I - (P_\lambda P_\mu)^k, \chi_{\wedge \Pi}] = 0. \quad (33)$$

Taking the limit $k \rightarrow \infty$ in (33), one notices that $\chi_{\wedge \Pi}$ is a projective operator, and thus

$$\neg \chi_{\wedge \Pi} = I - \chi_{\wedge \Pi}. \quad (34)$$

Combining (33) and (34), one obtains that $\forall k > 0$

$$\begin{aligned} I - (P_\lambda P_\mu)^k &= \neg \chi_{\wedge \Pi}, \\ (P_\lambda P_\mu)^k &= I - \neg \chi_{\wedge \Pi}, \\ H_C((P_\lambda P_\mu)^k) &= H_C(I - \neg \chi_{\wedge \Pi}) = H_C(\chi_{\wedge \Pi}). \end{aligned} \quad (35)$$

□

One should keep in mind that generally $(P_\lambda P_\mu)^k$ can not be considered as a projective operator until one takes the limit $k \rightarrow \infty$, and thus “=” in (35) should be rather considered as a symbol of logical equivalence (i.e. operators on both sides have similar logical properties), but not as a symbol of equality. As a result, one can not write $(P_\lambda P_\mu)^k = \chi_{\wedge_\Pi}$ for any finite $k > 0$ in general case.

7 Examples

Before we proceed, let us introduce some notations. At first we define the following projectors:

$$\begin{aligned} P_z &= |z\rangle\langle z|, \quad P_{\neg z} = I - P_z, & |z\rangle: \sigma_z|z\rangle &= |z\rangle, \\ P_x &= |x\rangle\langle x|, \quad P_{\neg x} = I - P_x, & |x\rangle: \sigma_x|x\rangle &= |x\rangle, \end{aligned}$$

where I is the unit operator, and σ_z, σ_x are the corresponding Pauli matrices. As it is known, so defined projectors P_z and P_x do not commute.

Using such formalism, one can write then

$$\begin{aligned} W_k &= \frac{1}{\sqrt{2}} (P_{zk} - P_{\neg zk} + P_{xk} - P_{\neg xk}) = \sqrt{2} (P_{xk} - P_{\neg zk}), \\ C_{k,s} &= (1 - e^{i\phi_{k,s}}) (P_{zs} P_{\neg zk} + I_s P_{zk}) + e^{i\phi_{k,s}} I_s I_k, \end{aligned}$$

where W_k is the Walsh-Hadamard gate on the k -th qubit, and $C_{k,s}$ is the controlled-phase gate on the k -th and s -th qubits with phase shift $\phi_{k,s} = \pi/2^{s-k}$. Here and in the following the operator's subscripts $_{k,s}$ denote the qubits these operators act on.

Now we can use (31) for estimation of the efficiency loss of any quantum algorithm \mathbb{E}_Q under the dequantization. Processing the estimation we should keep in mind that the number of conjunctions c_1 equals in the number of projector's intersections from the viewpoint of the efficiency loss (see Theorem 1).

7.1 FFT_Q dequantization

As it is known, FFT_Q on the N -qubit register may be written as the following operator:

$$\begin{aligned} \text{FFT}_Q &= \Phi_0 \cdots \Phi_{N-1}, \\ \Phi_k &= W_k C_{k,N-1} C_{k,N-2} \cdots C_{k,k+1}. \end{aligned}$$

One can notice that every $C_{k,s}$ contains 2 non-reducing terms with P_{zk} which do not commute with the corresponding P_{xk} of W_k . Then $c_{I_k} = 2^{N-k-1}$ for any Φ_k and

$$c_1 = \sum_{k=0}^{N-1} c_{I_k} = \sum_{k=0}^{N-1} 2^{N-k-1} = 2^N - 1. \quad (36)$$

Substituting (36) and $\dim \Pi_k = 2^N$ in (31), we obtain then

$$H_C(\text{FFT}_Q) \leq \left[q_1 + (N+1)(2^N - 1) \right] \ln 2 = \mathcal{O}(N2^N), \quad (37)$$

thus meeting an exponential loss of computational efficiency of the dequantized FFT_Q. As it is known, its classical analog FFT_C needs $\mathcal{O}(N2^N)$ amount of resources.

7.2 Grover dequantization

Gr_Q , which is working on the database containing 2^N elements, can be represented with the following operator

$$\begin{aligned}\text{Gr}_Q &= \left[D \otimes (P_{\neg z} - P_z) \right]^{\frac{\pi}{4} 2^{N/2}} U_\Gamma, \\ D &= 2(WP_z W)^{\otimes N} - I^{\otimes N}, \\ U_\Gamma &: |x\rangle|0\rangle \rightarrow |x\rangle|\Gamma(x)\rangle,\end{aligned}$$

where Γ is the tested statement (i.e. Gr_Q determines the elements on which Γ is true). The operator U_Γ requires a number of gates which depends on particular expression for Γ , and thus will not be considered in the following.

As for the component $(P_{\neg z} - P_z)$ acting on the ancillary qubit, it includes $c_{I_k|\Gamma} = 1$ intersections for the complementary (and hence commuting) projectors only, for which one can put formally $\dim \Pi_{k|\Gamma} = 1$ while estimating the efficiency loss. The number of these ancillary intersections is

$$c_{I|\Gamma} = \sum_{k=1}^{\frac{\pi}{4} 2^{N/2}} c_{I_k|\Gamma} = \frac{\pi}{4} 2^{N/2}. \quad (38)$$

Then one obtains for operator D that

$$WP_z W = 2(P_x - P_{\neg z})P_z(P_x - P_{\neg z}) = 2P_x P_z P_x,$$

thus giving one intersection of non-commuting projectors. The number of such intersections in the operator D is $c_{I_k|D} = N$ (one for every qubit in the register). As D should be applied $\frac{\pi}{4} 2^{N/2}$ times, then

$$c_{I|D} = \sum_{k=1}^{\frac{\pi}{4} 2^{N/2}} c_{I_k|D} = \sum_{k=1}^{\frac{\pi}{4} 2^{N/2}} N = \frac{\pi}{4} N 2^{N/2}. \quad (39)$$

As $\dim \Pi_{k|D} = 2^N$, one obtains after substituting (38) and (39) into (31)

$$\begin{aligned}H_C(\text{Gr}_Q) &\leq (q_I + c_{I|\Gamma}) \ln 2 + c_{I|D} (1 + N) \ln 2 \\ &= \left[q_I + \frac{\pi}{4} (N+1)^2 2^{N/2} \right] \ln 2 = \mathcal{O}(N^2 2^{N/2}).\end{aligned} \quad (40)$$

As it is known, classical search algorithm requires $\mathcal{O}(2^N)$ number of resources, while Gr_Q needs $\mathcal{O}(2^{N/2})$. Thus, as we see, in this case one obtains non-polynomial efficiency loss. It may be an example of the “not complete” algorithm reduction, i.e. when the algorithm under the dequantization reduces to the rather complicated one.

As the search algorithm belongs to the NP complexity class, the example demonstrates that at least some quantum algorithms being NP (here this is Gr_Q) do not meet complete efficiency loss (i.e. the efficiency loss for them does not necessarily equal in the number of resources required with their classical analogs) under dequantization: here we obtained $\mathcal{O}(N^2 2^{N/2})$ instead of $\mathcal{O}(2^N)$ efficiency loss. The reason for such a difference is the transformation of the algorithm which is discussed in the next section.

8 Discussion

In the manuscript we dequantized the complete set of elementary quantum logic operations including the non-commuting conjunction. We calculated the efficiency loss and an upper bound on it for the operations under the dequantization. This allowed to derive the general expression estimating the loss of computational efficiency for any dequantized quantum algorithm, see (30) and (31). We formulated and proved the theorem needed for estimation of conjunction of non-commuting projectors. Finally, the obtained results were applied for estimation of the efficiency loss for FFT_Q and Gr_Q algorithms. The developed method demonstrated exponential (37) and non-polynomial (40) efficiency loss for the algorithms correspondingly.

As it was already noticed, the Kolmogorov complexity approach is used to estimate the difference between quantum and classical calculation. It gives the minimized in size program that realizes the corresponding algorithm. Such an approach helps to define conditions on the calculations, which are easy in quantum but are hard in classical case. For more details see P. Vitanyi [1] and other papers such as [3, 28, 10].

In case of the algorithmic entropies [20], they may be used to describe “distance” between the desired and the calculated result in case of using quantum or classical algorithm.

However, our approach seems to differ from the previous. Dequantization of elementary QL operations allows to estimate the corresponding entropy for any logical expression. It gives the amount of information loss during the *reduction* of quantum algorithm to the classical one. It has much in common with, but can not be interpreted as comparison of the corresponding Kolmogorov complexities for the quantum and classical algorithms solving *the same* problem. The number of elementary operations is the same before and after dequantization. The similarity origins from the re-estimating quantum gates in the number of classical ones. But, after the dequantization algorithm may solve another problem (like FFT_Q), thus pointing out the differences with the Kolmogorov approach.

Also our approach seems not to be similar to algorithm entropies. The entropies are used to estimate the probability of obtaining the desired result, while our approach is designed to estimate the *efficiency loss* of calculation under the dequantization.

As the illustration of this statement one can use the classical discrete fast Fourier transform (FFT). As it is known, FFT_Q is a polynomial time algorithm. It needs $\mathcal{O}(N^2 + N)$ operations, while FFT needs $\mathcal{O}(N2^N)$. According to section VII, the number of elementary operations during dequantization remains the same, i.e. polynomial. However, some amount of information is lost, and this amount can be estimated. It seems that the only reasonable explanation for this is the algorithm changeover. For this particular case, FFT_Q transforms into the Legendre transform (but not into FFT), as it was shown in [19]; for some more information one can see [35].

Such algorithm simplification after dequantization is explained by the fact that QL algebra can be splitted into some Boolean subalgebras, each of which is similar to the CL algebra, see [34]. The statements from different subalgebras do not commute, thus providing the largest possible loss of computational efficiency.

Summarizing we conclude that the main reason for the efficiency loss under dequantization is the presence of non-commuting statements in the quantum logical expression \mathbb{E}_I . The entanglement itself seems to play not such a significant role in providing the efficiency loss; one can find more on the topic in [24].

Some questions still remain open, and it seems reasonable to solve them. There are few of them:

1. If some optimal quantum algorithm gives an exponential information loss, whether it implies, that the classical algorithm for the same problem is exponential in time?
2. Can the presented approach be used for the comparison of quantum and classical algorithm complexity classes, and correlation ascertainment between these classes?
3. In general, under dequantization quantum algorithm transforms into another one, that solves another problem. But, what about the reversion: can one obtain some quantum algorithm (or the class of them), being given the classical one? Some investigation on the topic of transition from subsets of CL to compatible (i.e. determined with mutually commuting operator sets) linear subspaces of QL one can find in [14]; it is called lifting in the manuscript. On our opinion, such a reversion should be ambiguous due to the differences between subsets and linear subspaces. The point is that there is no recipe to go to the incompatible subspaces. On our opinion, formalization and further development of the approach presented in [31] might be helpful while constructing the recipe. However it is not known for sure whether it can be solved or not at least for some classes of quantum algorithms. In particular one can try to build the quantum analog of the Legendre transform. Due to ambiguity one is expected to derive some class of quantum algorithms but not FFT_Q only. One more interesting point is to look for the quantum analogs of inefficient classical algorithms such as FFT or factorization and to verify whether the analogs will be inefficient in QL too.

Summing up the questions mentioned above we conclude that the problem of the reverse transition to dequantization is worth of further investigation.

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References

1. 15th IEEE Conf. Computational Complexity: Three Approaches to the Quantitative Definition of Information in an Individual Pure Quantum State. IEEE Conf. Computational Complexity (2000)
2. 19th IEEE conference on Logic in Computer Science (LiCS'04): A categorical semantics of quantum protocols. IEEE Computer Science Press (2004)
3. A Berthiaume, W.v.D., Laplante, S.: Quantum kolmogorov complexity. J. of Computer and Systems Sciences **63**(2), 201–221 (2001)
4. Abramsky, S., Duncan, R.: A categorical quantum logic. arXiv:quant-ph/0512114 (2005)
5. Alvarez, E.T.G.: The logic behind feynman's paths. arXiv:1011.4971 [quant-ph] (2010)
6. Battilotti, G., Zizzi, P.: Logical interpretation of a reversible measurement in quantum computing. arXiv:quant-ph/0408068 (2004)

7. Benadives, J.: Sheaf logic, quantum set theory and the interpretation of quantum mechanics. arXiv:1111.2704 [math-ph] (2011)
8. Birkhoff, G., Neumann, J.: The logic of quantum mechanics. *Ann. Math.* **37**(4), 823–843 (1936)
9. Brunet, O.: A rule-based logic for quantum information. arXiv:cs/0504018 [cs.LO] (2005)
10. C. Mora C., H.B., Kraus, B.: Quantum kolmogorov complexity and its applications. arXiv:quant-ph/0610109 (2006)
11. Chiara, M.L.D., Giuntini, R.: Quantum logic. arXiv:quant-ph/0101028 (2001)
12. D Lehmann, K.E., Gabbay, D.M.: Algebras of measurements: the logical structure of quantum mechanics. *Int. J. Theor. Phys.* **45**(4), 698–723 (2006)
13. Domenech, G., Freytes, H.: Contextual logic for quantum systems. *J. Math. Phys.* **46**, 012,102 (2005)
14. Ellerman, D.: The objective indefiniteness interpretation of quantum mechanics. arXiv:1210.7659 [quant-ph] (2012)
15. F Holik, C.M., Ciancaglini, N.: Convex quantum logic. arXiv:1008.4168 [quant-ph] (2010)
16. G Domenech, F.H., Massri, C.: A quantum logical and geometrical approach to the study of improper mixtures. *J. Math. Phys.* **51**, 052,108 (2010)
17. G Domenech, H.F., Ronde, C.: Scopes and limits of modality in quantum mechanics. *Ann. Phys. (Leipzig)* **15**, 853–860 (2006)
18. G Domenech, H.F., Ronde, C.: A topological study of contextuality and modality in quantum mechanics. *Int. J. Theor. Phys.* **47**, 168–174 (2008)
19. G Litvinov, V.M., Shpiz, G.: Idempotent (asymptotic) mathematics and the representation theory. arXiv:math/0206025 [math.RT] (2002)
20. Gacs, P.: Quantum algorithmic entropy. *J. Phys. A: Math. Gen.* **34**, 1–22 (2001)
21. Garola, C.: A pragmatic interpretation of quantum logic. arXiv:quant-ph/0507122 (2005)
22. Garola, C.: Physical propositions and quantum languages. *Int. J. Theor. Phys.* **47**(1), 90–103 (2008)
23. Isham, C.J.: Quantum logic and decohering histories. arXiv:quant-ph/9506028 (1995)
24. Kendon, V.M., Munro, W.J.: Entanglement and its role in shor’s algorithm. *Quantum Info. & Computation* **6**(7), 630–640 (2006)
25. Lehmann, D.: A presentation of quantum logic based on an “and then” connective. *J. Logic and Comput.* **18**(1), 59–76 (2008)
26. M L D Chiara, R.G., Leporini, R.: Quantum computational logics. a survey. arXiv:quant-ph/0305029 (2003)
27. Marchetti, P.A., Rubele, R.: Quantum logic and non-commutative geometry. *Int. J. Phys.* **46**, 49–62 (2007)
28. Mora, C., Briegel, H.J.: Algorithmic complexity of quantum states. arXiv:quant-ph/0412172 (2004)
29. Nest, M.V., Briegel, H.J.: Measurement-based quantum computation and undecidable logic. *Found. Phys.* **38**(5), 448–457 (2008)
30. Neumann, J.V.: *Mathematische Grundlagen der Quantenmechanik*. Springer, Berlin (1932)
31. Nicolaidis, A.: Relational quantum mechanics (2012)
32. P D Bruza, D.W., Woods, J.: *Handbook of Quantum Logic, Quantum Structure and Quantum Computation*, vol. 2, chap. A Quantum Logic of Down Below (2006)
33. Papaikolaou, N.: Logic column 13: Reasoning formally about quantum systems: An overview. *SIGACT News* **36**(3), 51–66 (2005)
34. Svozil, K.: *Handbook of Quantum Logic and Quantum Structures*. Elsevier, Amsterdam (2008)
35. T Yajima, K.N., Asano, N.: Max-plus algebra for complex variables and its application to discrete fourier transformation. *J. Phys. Soc. Jpn.* **75**, 064,001 (2006)
36. Tonder, A.: A lambda calculus for quantum computation. *SIAM J. Comput.* **33**, 1109–1135 (2004)
37. Ying, M.: A theory of computation based on quantum logic(i). arXiv:cs/0403041 [cs.LO] (2004)
38. Zizzi, P.: Basic logic and quantum entanglement. *J. Phys. Conf. Ser.* **67**, 012,045 (2007)